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Ann Brett

University of Rhode Island, abrett@uri.edu

E. J. Janowski

University of Rhode Island

See next page for additional authors

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Authors

Ann Brett, E. J. Janowski, and Mustafa Kulenovic

Research Article

Global Asymptotic Stability for Linear Fractional Difference Equation

A. Brett, E. J. Janowski, and M. R. S. Kulenović

Department of Mathematics, University of Rhode Island, Kingston, RI 02881-0816, USA

Correspondence should be addressed to M. R. S. Kulenović; kulenm@math.uri.edu

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Consider the difference equation $x_{n+1} = (\alpha + \sum_{i=0}^k a_i x_{n-i}) / (\beta + \sum_{i=0}^k b_i x_{n-i})$, $n = 0, 1, \dots$, where all parameters α, β, a_i, b_i , $i = 0, 1, \dots, k$, and the initial conditions x_i , $i \in \{-k, \dots, 0\}$ are nonnegative real numbers. We investigate the asymptotic behavior of the solutions of the considered equation. We give easy-to-check conditions for the global stability and global asymptotic stability of the zero or positive equilibrium of this equation.

1. Introduction

Consider the difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^k a_i x_{n-i}}{\beta + \sum_{i=0}^k b_i x_{n-i}}, \quad n = 0, 1, \dots, \quad (1)$$

where $k \in \{1, \dots\}$, the parameters α, β, a_i, b_i , $i = 0, 1, \dots, k$, and the initial conditions x_i , $i \in \{-k, \dots, 0\}$ are nonnegative real numbers. The important special cases of (1) are the well-known Riccati equation

$$x_{n+1} = \frac{\alpha + a_0 x_n}{\beta + b_0 x_n}, \quad n = 0, 1, \dots, \quad (2)$$

the second order linear fractional difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^1 a_i x_{n-i}}{\beta + \sum_{i=0}^1 b_i x_{n-i}}, \quad n = 0, 1, \dots, \quad (3)$$

and the third order linear fractional difference equation that we get from (1) for $k = 2$. The global behavior and the exact solutions of (2) even for real parameters have been found in [1]. The global behavior of solutions of (3), in many subcases when one or more parameters are zero, was established in [1]. There are still some conjectures left whose answers will complete the global picture of the asymptotic behavior for the solutions of (3). As far as the third order linear fractional

difference equation is concerned, there are a large number of sporadic results that are systemized in a book [2]. The characterization of the global asymptotic behavior of the solutions of (1) for $k = 2$ seems to be much harder than for the second order equation (3). Consequently an attempt at giving the characterization of the global asymptotic behavior for the solutions of (1) seems to be a formidable task at this time. However using some known global attractivity results we can describe the global asymptotic behavior for the solutions of (1) in some subspaces of the parametric space and the space of initial conditions. See [2–6] for a complete description of the behavior of some special cases of (1), in particular for the cases known as periodic trichotomies. See [7] where the difference in global behavior between the second and third order linear fractional difference equation is emphasized. The results on the global periodicity, that is, the results which describe all special cases of (1) where all solutions are periodic of the same period, were obtained in [8, 9]. Most results in [2–6, 10, 11] are based on known global attractivity or global asymptotic stability results obtained in [1, 2, 12–17].

This paper is an attempt at establishing some global stability results for the equilibrium solution(s) of (1). Our results give effective conditions for global asymptotic stability of the equilibrium solution(s) of (1) expressed in terms of the inequalities on the coefficients. It is worth mentioning that the long standing conjecture for (3) is that local asymptotic stability implies global asymptotic stability of the equilibrium

[2, 4]. In the case of the third order equation (1) with $k = 2$, the standing conjecture is that local asymptotic stability and boundedness of all solutions imply global asymptotic stability of the equilibrium. If the second conjecture is proved to hold, it will still be very difficult to verify the conditions for local asymptotic stability of the equilibrium as these conditions are very difficult to check for linear fractional equations of order higher than 2. See [2] for many special cases of third order linear fractional equation with very complicated conditions for local asymptotic stability. Thus the presented results are of importance even if the abovementioned conjecture is proved to be true.

The following general global results will be applied to (1); see [18]. Consider the difference equation

$$x_{n+1} = f(x_n, \dots, x_{n-k}), \quad n = 0, 1, \dots, \quad (4)$$

where $k \in \{0, 1, \dots\}$. Sometimes it is more advantageous to investigate (4) by embedding (4) into a higher iteration of the form

$$x_{n+j} = F_j(x_{n+j-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \quad (5)$$

where $j \in \{2, 3, \dots\}$ (see [16, 18, 19]) and then linearizing (4) or (5) by rewriting them (see [18]) into a nonautonomous linear equation of the form

$$x_{n+l} = \sum_{i=1-l}^k g_i x_{n-i}, \quad n = 0, 1, \dots, \quad (6)$$

where $l \in \{1, 2, \dots\}$ and the functions $g_i : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ are in general functions of both n and the state variables x_i , $i = n - k, \dots, n + l - 1$. See [18, 20] for examples of such linearizations.

Theorem 1. Let $l \in \{1, 2, \dots\}$. Suppose that (4) has the linearization (6) where the functions $g_i : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ are such that

$$\sum_{i=1-l}^k |g_i| \leq a < 1, \quad n = 0, 1, \dots \quad (7)$$

Then

$$\lim_{n \rightarrow \infty} x_n = 0. \quad (8)$$

As we have observed in [18], condition (7) is actually a contraction condition in the Banach contraction principle.

In addition, we will need the following stability result which is a consequence of our results in [18].

Theorem 2. Suppose that (4) can be linearized into the form

$$x_{n+1} - \bar{x} = \sum_{i=0}^k g_i (x_{n-i} - \bar{x}), \quad n = 0, 1, \dots, \quad (9)$$

where \bar{x} is an equilibrium of (4) and the functions $g_i : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$. If $\sum_{i=0}^k |g_i| \leq 1$, $n \geq 0$, then the equilibrium \bar{x} of (4) is stable.

Proof. Observe that

$$|x_{n+1} - \bar{x}| \leq \sum_{i=0}^k |g_i| |x_{n-i} - \bar{x}|, \quad n = 0, 1, \dots \quad (10)$$

Assume that $\sum_{i=0}^k |x_{n-i} - \bar{x}| < \delta$. Take $\varepsilon = \delta$. Then (9) implies

$$|x_1 - \bar{x}| \leq \sum_{i=0}^k |g_i| |x_{-i} - \bar{x}| < \delta \sum_{i=0}^k |g_i| \leq \delta, \quad (11)$$

$$|x_2 - \bar{x}| \leq \sum_{i=0}^k |g_i| |x_{1-i} - \bar{x}| < \delta \sum_{i=0}^k |g_i| \leq \delta,$$

and so by induction $|x_n - \bar{x}| < \delta = \varepsilon$ for $n \geq -k$. \square

2. Preliminaries

First observe that when $\sum_{i=0}^k b_i = 0$ (1) becomes the linear nonhomogeneous equation

$$x_{n+1} = \frac{\alpha}{\beta} + \sum_{i=0}^k g_i x_{n-i}, \quad n = 0, 1, \dots, \quad (12)$$

where $g_i = a_i/\beta$ for all $i = 0, 1, \dots, k$ and whose equilibrium \bar{x} satisfies $\beta\bar{x} = \alpha + \bar{x} \sum_{i=0}^k a_i$.

We now establish our first result.

Theorem 3. Let $\beta > 0$ and $\sum_{i=0}^k b_i = 0$.

- (1) If $\alpha = 0$ and $\beta > \sum_{i=0}^k a_i$, then the zero equilibrium of (1) is globally asymptotically stable.
- (2) If $\alpha = 0$ and $\beta = \sum_{i=0}^k a_i$, then the zero equilibrium of (1) is stable.
- (3) If $\alpha = 0$ and $\beta < \sum_{i=0}^k a_i$, then $\lim_{n \rightarrow \infty} x_n = \infty$ whenever, for some $n \geq 0$, $x_{n-i} > 0$, $i = 0, \dots, k$.
- (4) If $\alpha > 0$ and $\beta > \sum_{i=0}^k a_i$, then the unique positive equilibrium of (1) is globally asymptotically stable.

Proof. When $\alpha = 0$ and $\sum_{i=0}^k b_i = 0$, (1) becomes

$$x_{n+1} = \sum_{i=0}^k \frac{a_i}{\beta} x_{n-i}, \quad n = 0, 1, \dots \quad (13)$$

- (1) In this case $\sum_{i=0}^k |g_i| = \sum_{i=0}^k (a_i/\beta) < 1$ and the result follows from Theorems 1 and 2.
- (2) In this case $\sum_{i=0}^k |g_i| = \sum_{i=0}^k (a_i/\beta) = 1$ and the result follows from Theorem 2.
- (3) Since $g_i \geq 0$, $i \in \{0, \dots, k\}$ and $\sum_{i=0}^k g_i = \sum_{i=0}^k (a_i/\beta) > 1$, then the result follows from Theorem 2 in [18].

When $\alpha > 0$ and $\sum_{i=0}^k b_i = 0$, (1) becomes

$$x_{n+1} = \frac{\alpha}{\beta} + \sum_{i=0}^k \frac{a_i}{\beta} x_{n-i}, \quad n = 0, 1, \dots, \quad (14)$$

and has a unique positive equilibrium \bar{x} provided $\beta > \sum_{i=0}^k a_i$. Then

$$\begin{aligned} x_{n+1} - \bar{x} &= \sum_{i=0}^k \frac{a_i}{\beta} (x_{n-i} - \bar{x}) + \frac{\bar{x} \sum_{i=0}^k a_i + \alpha - \beta \bar{x}}{\beta} \\ &= \sum_{i=0}^k \frac{a_i}{\beta} (x_{n-i} - \bar{x}). \end{aligned} \quad (15)$$

Let $y_n = x_n - \bar{x}$ where $n \geq 0$, $\bar{x} > 0$. Then, for $n \geq 0$, y_n satisfies

$$y_{n+1} = \sum_{i=0}^k \frac{a_i}{\beta} y_{n-i}. \quad (16)$$

- (4) Since $\sum_{i=0}^k |g_i| = \sum_{i=0}^k (a_i/\beta) < 1$, then, by Theorem 1, $\lim_{n \rightarrow \infty} y_n = 0$ and so $\lim_{n \rightarrow \infty} x_n = \bar{x}$. Thus \bar{x} is a global attractor. By applying Theorem 2 to (15) we get that the equilibrium \bar{x} is stable and so the positive equilibrium of (1) is globally asymptotically stable. \square

For the remainder of this paper we will assume that $b_i > 0$ for at least one $i \in \{0, \dots, k\}$.

Now we investigate the stability of the zero equilibrium of (1). Note that (1) has a zero equilibrium if and only if $\alpha = 0$ and $\beta \neq 0$.

Theorem 4. Let $\beta > 0$ and $\sum_{i=0}^k b_i > 0$. Assume that $\alpha = 0$.

- (1) If $\beta > \sum_{i=0}^k a_i > 0$, then the zero equilibrium of (1) is globally asymptotically stable.
- (2) If $\beta = \sum_{i=0}^k a_i > 0$, then the zero equilibrium of (1) is stable.

Proof. When $\alpha = 0$ (1) becomes

$$x_{n+1} = \frac{\sum_{i=0}^k a_i x_{n-i}}{\beta + \sum_{i=0}^k b_i x_{n-i}}, \quad n = 0, 1, \dots, \quad (17)$$

which can be written in the linearized form (6) where $l = 1$:

$$x_{n+1} = \sum_{i=0}^k \frac{a_i}{\beta + \sum_{i=0}^k b_i x_{n-i}} x_{n-i}, \quad n = 0, 1, \dots \quad (18)$$

Define $h_i = a_i/(\beta + \sum_{i=0}^k b_i x_{n-i})$, $n \geq 0$ for $i = 0, \dots, k$.

- (1) The proof follows from Theorems 1 and 2 as $\sum_{i=0}^k |h_i| = \sum_{i=0}^k (a_i/(\beta + \sum_{i=0}^k b_i x_{n-i})) \leq \sum_{i=0}^k a_i/\beta < 1$.
- (2) The proof follows from Theorem 2 as $\sum_{i=0}^k |h_i| = \sum_{i=0}^k (a_i/(\beta + \sum_{i=0}^k b_i x_{n-i})) \leq \sum_{i=0}^k a_i/\beta = 1$. \square

3. Positive Equilibrium

In this section we investigate the stability of the unique positive equilibrium of (1) by using Theorems 1 and 2.

Note that, for $n = 0, 1, \dots$, the function

$$f(x_n, \dots, x_{n-k}) = \frac{\alpha + \sum_{i=0}^k a_i x_{n-i}}{\beta + \sum_{i=0}^k b_i x_{n-i}} \quad (19)$$

has the following properties:

- (a) if $a_i > 0$ and $b_i = 0$, then f is increasing in x_{n-i} on the interval $[0, \infty)$;
- (b) if $a_i = 0$ and $b_i > 0$, then f is decreasing in x_{n-i} on the interval $[0, \infty)$.

The following result gives some other cases when f is monotonic.

Remark 5. Consider the function f given by (19) on $[0, \infty)^{k+1}$ where \bar{x} is a unique positive fixed point of this function. Assume that $a_i, b_i > 0$ for $i \in \{0, \dots, k\}$. Then for $k \in \{0, 1, \dots\}$ set

$$f(u_0, \dots, u_k) = \frac{\alpha + \sum_{i=0}^k a_i u_i}{\beta + \sum_{i=0}^k b_i u_i}, \quad u_i \in [0, \infty), \quad i = 0, \dots, k. \quad (20)$$

Then for $i \in \{0, \dots, k\}$

$$\begin{aligned} f_{u_i} &= \frac{\partial f}{\partial u_i} = \frac{(\beta + \sum_{j=0}^k b_j u_j) a_i - (\alpha + \sum_{j=0}^k a_j u_j) b_i}{(\beta + \sum_{j=0}^k b_j u_j)^2} \\ &= b_i \frac{a_i/b_i - f(u_0, \dots, u_k)}{\beta + \sum_{j=0}^k b_j u_j}, \end{aligned} \quad (21)$$

$$f_{u_i}(\bar{x}, \dots, \bar{x}) = \frac{a_i - \bar{x} b_i}{\beta + \bar{x} \sum_{j=0}^k b_j}.$$

Thus we have that, for $i \in \{0, \dots, k\}$, $f_{u_i} > 0$ on the interval $[0, a_i/b_i)$ and $f_{u_i} < 0$ on the interval $(a_i/b_i, \infty)$.

In the case when f is monotonic in all its arguments one can try to use global attractivity and global asymptotic stability results established in [1, 2, 15, 16].

In order to apply Theorem 1 to (1) we first need to linearize (1) into the form (6) which can be done as follows:

$$\begin{aligned} x_{n+1} - \bar{x} &= \frac{\sum_{i=0}^k a_i (x_{n-i} - \bar{x})}{\beta + \sum_{i=0}^k b_i x_{n-i}} \\ &\quad + \frac{\alpha + \bar{x} \sum_{i=0}^k a_i - \beta \bar{x} - \bar{x} \sum_{i=0}^k b_i x_{n-i}}{\beta + \sum_{i=0}^k b_i x_{n-i}}, \end{aligned} \quad (22)$$

$n = 0, 1, \dots$

Now applying the equilibrium equation we get that for $n \geq 0$

$$\begin{aligned} x_{n+1} - \bar{x} &= \frac{\sum_{i=0}^k a_i (x_{n-i} - \bar{x})}{\beta + \sum_{i=0}^k b_i x_{n-i}} - \frac{\bar{x} \sum_{i=0}^k b_i (x_{n-i} - \bar{x})}{\beta + \sum_{i=0}^k b_i x_{n-i}} \\ &= \sum_{i=0}^k \frac{a_i - \bar{x} b_i}{\beta + \sum_{i=0}^k b_i x_{n-i}} (x_{n-i} - \bar{x}). \end{aligned} \quad (23)$$

Let $y_n = x_n - \bar{x}$ for $n \geq 0$ and $\bar{x} > 0$. Then y_n satisfies

$$y_{n+1} = \sum_{i=0}^k g_i y_{n-i}, \quad n = 0, 1, \dots, \quad (24)$$

where for $i = 0, \dots, k$

$$g_i = \frac{a_i - \bar{x} b_i}{\beta + \sum_{i=0}^k b_i x_{n-i}}, \quad n = 0, 1, \dots \quad (25)$$

The conditions $a_i > b_i \bar{x}$, $a_i < b_i \bar{x}$ and $a_i = b_i \bar{x}$, which are equivalent to $g_i > 0$, $g_i < 0$, and $g_i = 0$, can be reformulated in a more explicit way.

Proposition 6. Let $a_j, b_j > 0$ for some $j \in \{0, \dots, k\}$ and let \bar{x} be the positive equilibrium of (1). Then for $j \in \{0, \dots, k\}$

- (a) $a_j > b_j \bar{x}$ if and only if $a_j^2 \sum_{i=0}^k b_i + a_j b_j (\beta - \sum_{i=0}^k a_i) > \alpha b_j^2$,
- (b) $a_j < b_j \bar{x}$ if and only if $a_j^2 \sum_{i=0}^k b_i + a_j b_j (\beta - \sum_{i=0}^k a_i) < \alpha b_j^2$,
- (c) $a_j = b_j \bar{x}$ if and only if $a_j^2 \sum_{i=0}^k b_i + a_j b_j (\beta - \sum_{i=0}^k a_i) = \alpha b_j^2$.

Proof. Consider the following.

Case 1 ($\alpha = 0$). In this case (1) has the positive equilibrium $\bar{x} = (\sum_{i=0}^k a_i - \beta) / \sum_{i=0}^k b_i$ provided $\sum_{i=0}^k a_i > \beta$. Now case (a) becomes

$$a_j > \frac{(\sum_{i=0}^k a_i) - \beta}{\sum_{i=0}^k b_i} b_j \quad (26)$$

if and only if $a_j \sum_{i=0}^k b_i > (\sum_{i=0}^k a_i - \beta) b_j$ which proves (a). The proofs of parts (b) and (c) are similar.

Case 2 ($\alpha > 0$). Then (1) has the positive equilibrium

$$\bar{x} = \frac{-(\beta - \sum_{i=0}^k a_i) + \sqrt{(\beta - \sum_{i=0}^k a_i)^2 + 4\alpha \sum_{i=0}^k b_i}}{2 \sum_{i=0}^k b_i}. \quad (27)$$

- (a) Assume that $a_j > b_j \bar{x}$. Then

$$a_j > b_j \left[\frac{-(\beta - \sum_{i=0}^k a_i) + \sqrt{(\beta - \sum_{i=0}^k a_i)^2 + 4\alpha \sum_{i=0}^k b_i}}{2 \sum_{i=0}^k b_i} \right] \quad (28)$$

implies

$$\begin{aligned} &\left[2a_j \sum_{i=0}^k b_i + b_j \left(\beta - \sum_{i=0}^k a_i \right) \right]^2 \\ &> b_j^2 \left[\left(\beta - \sum_{i=0}^k a_i \right)^2 + 4\alpha \sum_{i=0}^k b_i \right] \end{aligned} \quad (29)$$

which yields

$$\begin{aligned} &4a_j^2 \left(\sum_{i=0}^k b_i \right)^2 + 4a_j b_j \left(\beta - \sum_{i=0}^k a_i \right) \sum_{i=0}^k b_i \\ &+ b_j^2 \left(\beta - \sum_{i=0}^k a_i \right)^2 \\ &> b_j^2 \left(\beta - \sum_{i=0}^k a_i \right)^2 + 4\alpha b_j^2 \sum_{i=0}^k b_i \end{aligned} \quad (30)$$

and so

$$a_j^2 \sum_{i=0}^k b_i + a_j b_j \left(\beta - \sum_{i=0}^k a_i \right) > \alpha b_j^2. \quad (31)$$

Now assume that $a_j^2 \sum_{i=0}^k b_i + a_j b_j (\beta - \sum_{i=0}^k a_i) > \alpha b_j^2$. Then

$$\begin{aligned} &\left(2a_j \sum_{i=0}^k b_i + b_j \left(\beta - \sum_{i=0}^k a_i \right) \right)^2 \\ &> b_j^2 \left[\left(\beta - \sum_{i=0}^k a_i \right)^2 + 4\alpha \sum_{i=0}^k b_i \right] \end{aligned} \quad (32)$$

and so

$$\begin{aligned} &2a_j \sum_{i=0}^k b_i + b_j \left(\beta - \sum_{i=0}^k a_i \right) \\ &> b_j \sqrt{\left(\left(\beta - \sum_{i=0}^k a_i \right)^2 + 4\alpha \sum_{i=0}^k b_i \right)}. \end{aligned} \quad (33)$$

Otherwise, suppose that

$$\begin{aligned} &2a_j \sum_{i=0}^k b_i + b_j \left(\beta - \sum_{i=0}^k a_i \right) \\ &\leq b_j \sqrt{\left(\left(\beta - \sum_{i=0}^k a_i \right)^2 + 4\alpha \sum_{i=0}^k b_i \right)}. \end{aligned} \quad (34)$$

Since

$$a_j \left[2a_j \sum_{i=0}^k b_i + b_j \left(\beta - \sum_{i=0}^k a_i \right) \right] > \alpha b_j^2 > 0, \quad (35)$$

then

$$\begin{aligned}
 & \left[2a_j \sum_{i=0}^k b_i + b_j \left(\beta - \sum_{i=0}^k a_i \right) \right]^2 \\
 & \leq \left[2a_j \sum_{i=0}^k b_i + b_j \left(\beta - \sum_{i=0}^k a_i \right) \right] \\
 & \quad \times b_j \sqrt{\left(\beta - \sum_{i=0}^k a_i \right)^2 + 4\alpha \sum_{i=0}^k b_i} \\
 & \leq \left[b_j \sqrt{\left(\beta - \sum_{i=0}^k a_i \right)^2 + 4\alpha \sum_{i=0}^k b_i} \right]^2,
 \end{aligned} \tag{36}$$

which is a contradiction.

Therefore,

$$a_j > b_j \left[\frac{-(\beta - \sum_{i=0}^k a_i) + \sqrt{(\beta - \sum_{i=0}^k a_i)^2 + 4\alpha \sum_{i=0}^k b_i}}{2 \sum_{i=0}^k b_i} \right] \tag{37}$$

and so $a_j > \bar{x}b_j$.

(b) Similarly we can show that $a_j \geq \bar{x}b_j$ if and only if $a_j^2 \sum_{i=0}^k b_i + a_j b_j (\beta - \sum_{i=0}^k a_i) \geq \alpha b_j^2$ from which the result follows.

(c) Assume that $a_j = \bar{x}b_j$. Suppose that

$$a_j^2 \sum_{i=0}^k b_i + a_j b_j \left(\beta - \sum_{i=0}^k a_i \right) \neq \alpha b_j^2. \tag{38}$$

Then either $a_j^2 \sum_{i=0}^k b_i + a_j b_j (\beta - \sum_{i=0}^k a_i) > \alpha b_j^2$ or $a_j^2 \sum_{i=0}^k b_i + a_j b_j (\beta - \sum_{i=0}^k a_i) < \alpha b_j^2$ and so either $a_j > \bar{x}b_j$ or $a_j < \bar{x}b_j$ which are both contradictions. Thus

$$a_j^2 \sum_{i=0}^k b_i + a_j b_j \left(\beta - \sum_{i=0}^k a_i \right) = \alpha b_j^2. \tag{39}$$

Similarly we can show that $a_j^2 \sum_{i=0}^k b_i + a_j b_j (\beta - \sum_{i=0}^k a_i) = \alpha b_j^2$ implies $a_j = \bar{x}b_j$. \square

We can now obtain easy-to-check conditions which show when the positive equilibrium of (1) is globally asymptotically stable. We will then apply these conditions to various cases of (1).

Theorem 7. Let $\sum_{i=0}^k b_i > 0$. Assume that one of the following holds:

- (1) $\sum_{i=0}^k |a_i - \bar{x}b_i| < \beta$;
- (2) there exist $L, N > 0$ such that for every solution $\{x_n\}$ of (1) $x_n \geq L$ for all $n \geq N$ and $\sum_{i=0}^k |a_i - \bar{x}b_i| < \beta + L \sum_{i=0}^k b_i$, where $\beta \geq 0$.

Then the positive equilibrium \bar{x} of (1) is globally asymptotically stable on the interval $[0, \infty)$.

Proof. As we have seen (1) can be written in the form of the linearized equation (24), where the coefficients g_i are given as (25).

(1) Observe that for $n \geq 0$

$$\begin{aligned}
 \sum_{i=0}^k |g_i| &= \sum_{i=0}^k \frac{|a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{n-i}} \\
 &= \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{n-i}} \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta} < 1.
 \end{aligned} \tag{40}$$

Then by Theorem 1, $\lim_{n \rightarrow \infty} y_n = 0$ and so $\lim_{n \rightarrow \infty} x_n = \bar{x}$. Thus \bar{x} is a global attractor on the interval $[0, \infty)$.

From (23) we have that $g_i = (a_i - \bar{x}b_i)/(\beta + \sum_{i=0}^k b_i x_{n-i})$, for $i = 0, 1, \dots, k$. Then

$$\sum_{i=0}^k |g_i| = \sum_{i=0}^k \left| \frac{a_i - \bar{x}b_i}{\beta + \sum_{i=0}^k b_i x_{n-i}} \right| \leq \sum_{i=0}^k \frac{|a_i - \bar{x}b_i|}{\beta} < 1, \tag{41}$$

which by Theorem 2 implies that the equilibrium \bar{x} is stable.

(2) Assume that there exist $L, N > 0$ such that for every solution $\{x_n\}$ of (1) $x_n \geq L$ for all $n \geq N$. Then for $n \geq 0$

$$\begin{aligned}
 \sum_{i=0}^k |g_i| &= \sum_{i=0}^k \frac{|a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{n-i}} \\
 &= \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{n-i}} \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta + L \sum_{i=0}^k b_i} < 1
 \end{aligned} \tag{42}$$

and, so by Theorem 1, \bar{x} is a global attractor on the interval $[L, \infty)$. Since, by assumption, $[L, \infty)$ is an attracting interval, then \bar{x} is a global attractor on the interval $[0, \infty)$.

By Theorem 2 applied to (23), \bar{x} is stable. Consequently, \bar{x} is globally asymptotically stable on the interval $[0, \infty)$. \square

Many cases of (1) have some combination of $a_i < \bar{x}b_i$, $a_i > \bar{x}b_i$, and $a_i = \bar{x}b_i$. In view of this we will adopt the following

notations where $I_> = \{i \mid \text{such that } a_i > \bar{x}b_i\}$, $I_ = \{i \mid \text{such that } a_i = \bar{x}b_i\}$, and $I_< = \{i \mid \text{such that } a_i < \bar{x}b_i\}$:

$$\begin{aligned}
 A_S &= \sum_{i \in I_>} a_i = \text{the sum of all the } a_i\text{'s}, \\
 B_S &= \sum_{i \in I_>} b_i = \text{the sum of all the } b_i\text{'s}, \\
 &\quad \text{such that } a_i > \bar{x}b_i, \\
 A_N &= \sum_{i \in I_ = } a_i = \text{the sum of all the } a_i\text{'s}, \\
 B_N &= \sum_{i \in I_ = } b_i = \text{the sum of all the } b_i\text{'s}, \\
 &\quad \text{such that } a_i = \bar{x}b_i, \\
 A_R &= \sum_{i \in I_<} a_i = \text{the sum of all the } a_i\text{'s}, \\
 B_R &= \sum_{i \in I_<} b_i = \text{the sum of all the } b_i\text{'s}, \\
 &\quad \text{such that } a_i < \bar{x}b_i.
 \end{aligned} \tag{43}$$

Then $A_S + A_N + A_R = \sum_{i=0}^k a_i$ and $B_S + B_N + B_R = \sum_{i=0}^k b_i$. Also $A_S > \bar{x}B_S$, $A_N = \bar{x}B_N$, and $A_R < \bar{x}B_R$.

Before we apply Theorem 7 to various cases of (1) we establish the following useful lemma.

Lemma 8. Let $\bar{x}, \alpha, \beta, \sum_{i=0}^k b_i > 0$ and $L \geq 0$. Then

$$\begin{aligned}
 (a) \quad & \left(2\beta + L \sum_{i=0}^k b_i\right) \left(\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i\right) > \alpha \sum_{i=0}^k b_i \\
 & \text{iff } \beta + L \sum_{i=0}^k b_i > \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i,
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 (b) \quad & \left(2\beta + L \sum_{i=0}^k b_i\right) \left(\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i\right) = \alpha \sum_{i=0}^k b_i \\
 & \text{iff } \beta + L \sum_{i=0}^k b_i = \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i,
 \end{aligned} \tag{45}$$

$$\begin{aligned}
 (c) \quad & \left(2\beta + L \sum_{i=0}^k b_i\right) \left(\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i\right) < \alpha \sum_{i=0}^k b_i \\
 & \text{iff } \beta + L \sum_{i=0}^k b_i < \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i.
 \end{aligned} \tag{46}$$

Proof. Observe that in Proposition 6 for $j \in \{0, \dots, k\}$ a_j and b_j are positive real numbers. Thus by Proposition 6 with $a_j = \beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i$ and $b_j = \sum_{i=0}^k b_i$ we have that

$$(a) \quad \left(2\beta + L \sum_{i=0}^k b_i\right) \left(\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i\right) > \alpha \sum_{i=0}^k b_i \tag{47}$$

if and only if

$$\begin{aligned}
 & \left(\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i\right)^2 \sum_{i=0}^k b_i + \left(\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i\right) \\
 & \times \left(\beta - \sum_{i=0}^k a_i\right) \sum_{i=0}^k b_i > \alpha \left(\sum_{i=0}^k b_i\right)^2
 \end{aligned} \tag{48}$$

if and only if

$$\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i > \bar{x} \sum_{i=0}^k b_i. \tag{49}$$

Cases (b) and (c) follow similarly. \square

Theorem 9. Let $\beta, \sum_{i=0}^k b_i > 0$. Then the positive equilibrium \bar{x} of (1) is globally asymptotically stable on the interval $[0, \infty)$ provided one of the following holds:

- (1) $a_i = \bar{x}b_i$ for all $i \in \{0, \dots, k\}$;
- (2) $a_i \geq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $\alpha > 0$;
- (3) $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $2\beta^2 + 2\beta \sum_{i=0}^k a_i > \alpha \sum_{i=0}^k b_i > 0$;
- (4) for some $i, j \in \{0, \dots, k\}$, $a_i > \bar{x}b_i$, $a_j < \bar{x}b_j$ and $\alpha/\bar{x} + 2A_S - 2\bar{x}B_S < 2\beta$, where $\alpha \geq 0$.

Proof. The positive equilibrium \bar{x} of (1) satisfies

$$\beta - \frac{\alpha}{\bar{x}} = \sum_{i=0}^k a_i - \bar{x} \sum_{i=0}^k b_i. \tag{50}$$

- (1) Let $a_i = \bar{x}b_i$ for all $i \in \{0, \dots, k\}$. Then $\sum_{i=0}^k a_i = \bar{x} \sum_{i=0}^k b_i$ and so $\beta = \alpha/\bar{x}$. Then (1) becomes, for $n \geq 0$,

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^k a_i x_{n-i}}{\beta + \sum_{i=0}^k b_i x_{n-i}} = \frac{\beta \bar{x} + \sum_{i=0}^k \bar{x} b_i x_{n-i}}{\beta + \sum_{i=0}^k b_i x_{n-i}} = \bar{x}. \tag{51}$$

- (2) Let $a_i \geq \bar{x}b_i$ for $i \in \{0, \dots, k\}$. Then for $i \in \{0, \dots, k\}$ we have $|a_i - \bar{x}b_i| = a_i - \bar{x}b_i$. Thus

$$\sum_{i=0}^k |a_i - \bar{x}b_i| = \sum_{i=0}^k a_i - \bar{x} \sum_{i=0}^k b_i = \beta - \frac{\alpha}{\bar{x}} < \beta \tag{52}$$

and the result follows from Theorem 7.

- (3) Let $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$. Then for $i \in \{0, \dots, k\}$ we have $|a_i - \bar{x}b_i| = \bar{x}b_i - a_i$. By Lemma 8 with $L = 0$

$$\sum_{i=0}^k |a_i - \bar{x}b_i| = \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i < \beta \quad (53)$$

and the result follows from Theorem 7.

- (4) For some $i, j \in \{0, \dots, k\}$, $a_i > \bar{x}b_i$ and $a_j < \bar{x}b_j$. Then

$$\begin{aligned} \sum_{i=0}^k |a_i - \bar{x}b_i| &= (A_S - \bar{x}B_S) + (\bar{x}B_N - A_N) + (\bar{x}B_R - A_R) \\ &= (A_S - A_N - A_R) + \bar{x}(B_R + B_N - B_S) \\ &= 2\bar{x}B_S - 2A_S - (A_R + A_N - A_S) \\ &\quad + \bar{x}(B_R + B_N - B_S) - 2\bar{x}B_S + 2A_S \\ &= \bar{x}(B_R + B_N + B_S) - (A_R + A_N + A_S) \\ &\quad + 2A_S - 2\bar{x}B_S \\ &= \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i + 2A_S - 2\bar{x}B_S. \end{aligned} \quad (54)$$

Observe that $\bar{x}(B_R + B_N - B_S) - (A_R + A_N - A_S) \geq 0$. Since $\alpha/\bar{x} + 2A_S - 2\bar{x}B_S < 2\beta$, then

$$\begin{aligned} \sum_{i=0}^k |a_i - \bar{x}b_i| &= \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i + 2A_S - 2\bar{x}B_S \\ &= \frac{\alpha}{\bar{x}} - \beta + 2A_S - 2\bar{x}B_S < \beta \end{aligned} \quad (55)$$

and so the result follows from Theorem 7. \square

There are many cases of (1) when we can establish a lower bound for all the solutions of (1).

Remark 10. The results on boundedness of all solutions of (1) are well known; see [2, 19]. For instance, if for every $i \in \{0, \dots, k\}$ such that $b_i > 0$ we have $a_i > 0$, then the uniform lower bound L for all solutions $\{x_n\}$ of (1), for $n \geq 1$, is

$$L = \frac{\min \{\alpha, a_i \mid a_i > 0\}}{\max \{\beta, b_i \mid b_i > 0\}}. \quad (56)$$

On the other hand, if for every $i \in \{0, \dots, k\}$ such that $a_i > 0$ we have $b_i > 0$, then the uniform lower bound L for all solutions of (1), for $n \geq 1$, is

$$L = \frac{\min \{\alpha, a_i \mid a_i > 0\}}{\max \left\{ \beta + U \sum_{j, a_j=0} b_j, b_i \mid b_i > 0 \right\}}, \quad (57)$$

where

$$U = \frac{\max \{\alpha, a_i \mid a_i > 0\}}{\min \{\beta, b_i \mid b_i > 0\}}. \quad (58)$$

See Example 19.

The results of Theorem 9 can be extended for those cases of (1) which have a lower bound L for every solution of (1); see Remark 10.

Theorem 11. Let $\sum_{i=0}^k b_i > 0$ and let $L, N > 0$ be such that, for $n \geq N$, $x_n \geq L$ for every solution $\{x_n\}$ of (1).

(1) If $\beta > 0$ and either

- (a) $a_i \geq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $\alpha = 0$;
- (b) $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $(2\beta + L \sum_{i=0}^k b_i)(\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i) > \alpha \sum_{i=0}^k b_i > 0$; or
- (c) for some $i, j \in \{0, \dots, k\}$ $a_i > \bar{x}b_i$, $a_j < \bar{x}b_j$, and

$$\frac{\alpha}{\bar{x}} + 2A_S - 2\bar{x}B_S < 2\beta + L \sum_{i=0}^k b_i, \quad (59)$$

where $\alpha \geq 0$,

then the positive equilibrium \bar{x} of (1) is globally asymptotically stable on the interval $[0, \infty)$.

(2) If $\beta = 0$ and either

- (a) $a_i = \bar{x}b_i$ for all $i \in \{0, \dots, k\}$;
- (b) $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $0 < \alpha < \bar{x}L \sum_{i=0}^k b_i$; or
- (c) for some $i, j \in \{0, \dots, k\}$ $a_i > \bar{x}b_i$, $a_j < \bar{x}b_j$, and

$$\frac{\alpha}{\bar{x}} + 2A_S - 2\bar{x}B_S < L \sum_{i=0}^k b_i, \quad (60)$$

where $\alpha \geq 0$,

then the positive equilibrium \bar{x} of (1) is globally asymptotically stable on the interval $(0, \infty)$.

Proof. (1) Assume that $\beta > 0$.

- (a) Let $a_i \geq \bar{x}b_i$ for $i \in \{0, \dots, k\}$. Since $\alpha = 0$, then

$$\sum_{i=0}^k |a_i - \bar{x}b_i| = \sum_{i=0}^k a_i - \bar{x} \sum_{i=0}^k b_i = \beta < \beta + L \sum_{i=0}^k b_i \quad (61)$$

and the result follows from Theorem 7.

- (b) Let $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$. By Lemma 8 with $L > 0$ we get

$$\sum_{i=0}^k |a_i - \bar{x}b_i| = \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i < \beta + L \sum_{i=0}^k b_i \quad (62)$$

and the result follows from Theorem 7.

(c) Assume that for some $i, j \in \{0, \dots, k\}$ $a_i > \bar{x}b_i$ and $a_j < \bar{x}b_j$. Then

$$\begin{aligned} \sum_{i=0}^k |a_i - \bar{x}b_i| &= \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i + 2A_S - 2\bar{x}B_S \\ &= \frac{\alpha}{\bar{x}} - \beta + 2A_S - 2\bar{x}B_S \\ &< \beta + L \sum_{i=0}^k b_i \end{aligned} \quad (63)$$

and the result follows from Theorem 7.

(2) Assume that $\beta = 0$.

(a) Let $a_i = \bar{x}b_i$ for all $i \in \{0, \dots, k\}$. Then $\alpha = 0$ and (1) becomes $x_{n+1} = \bar{x}$.

(b) Let $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$. Then

$$\sum_{i=0}^k |a_i - \bar{x}b_i| = \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i = \frac{\alpha}{\bar{x}} < L \sum_{i=0}^k b_i \quad (64)$$

and the result follows from Theorem 7.

(c) Assume that for some $i, j \in \{0, \dots, k\}$ $a_i > \bar{x}b_i$ and $a_j < \bar{x}b_j$. Then

$$\begin{aligned} \sum_{i=0}^k |a_i - \bar{x}b_i| &= \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i + 2A_S - 2\bar{x}B_S \\ &= \frac{\alpha}{\bar{x}} + 2A_S - 2\bar{x}B_S < L \sum_{i=0}^k b_i \end{aligned} \quad (65)$$

and the result follows from Theorem 7. \square

By using Theorem 2 and similar methods as in the proof of Theorems 7, 9, and 11 we can obtain the conditions for the stability of the positive equilibrium.

Theorem 12. Let $\sum_{i=0}^k b_i > 0$. Assume that one of the following holds:

- (1) $\sum_{i=0}^k |a_i - \bar{x}b_i| = \beta$;
- (2) there exist $L, N > 0$ such that for every solution $\{x_n\}$ of (1) $x_n \geq L$ for all $n \geq N$ and $\sum_{i=0}^k |a_i - \bar{x}b_i| = \beta + L \sum_{i=0}^k b_i$, where $\beta \geq 0$.

Then the positive equilibrium \bar{x} of (1) is stable on the interval $[0, \infty)$.

Proof. (1) Observe that for $n \geq 0$

$$\begin{aligned} \sum_{i=0}^k |g_i| &= \sum_{i=0}^k \frac{|a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{n-i}} \\ &= \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{n-i}} \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta} = 1. \end{aligned} \quad (66)$$

Thus the result follows from Theorem 2.

(2) The result follows similarly as in Theorem 7 part (2). \square

Theorem 13. Let $\beta, \sum_{i=0}^k b_i > 0$. Then the positive equilibrium \bar{x} of (1) is stable on the interval $[0, \infty)$ provided one of the following holds:

- (1) $a_i \geq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $\alpha = 0$;
- (2) $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $2\beta^2 + 2\beta \sum_{i=0}^k a_i = \alpha \sum_{i=0}^k b_i > 0$;
- (3) for some $i, j \in \{0, \dots, k\}$, $a_i > \bar{x}b_i$, $a_j < \bar{x}b_j$, and $\alpha/\bar{x} + 2A_S - 2\bar{x}B_S = 2\beta$ where $\alpha \geq 0$.

Theorem 14. Let $\sum_{i=0}^k b_i > 0$ and let $L, N > 0$ be such that for every solution $\{x_n\}$ of (1) $x_n \geq L$ for $n \geq N$.

(1) If $\beta > 0$ and either

- (a) $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $(2\beta + L \sum_{i=0}^k b_i)(\beta + \sum_{i=0}^k a_i + L \sum_{i=0}^k b_i) = \alpha \sum_{i=0}^k b_i > 0$ or
- (b) for some $i, j \in \{0, \dots, k\}$ $a_i > \bar{x}b_i$, $a_j < \bar{x}b_j$, and

$$\frac{\alpha}{\bar{x}} + 2A_S - 2\bar{x}B_S = 2\beta + L \sum_{i=0}^k b_i, \quad (67)$$

where $\alpha \geq 0$,

then the positive equilibrium \bar{x} of (1) is stable on the interval $[0, \infty)$.

(2) If $\beta = 0$ and either

- (a) $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and $0 < \alpha = \bar{x}L \sum_{i=0}^k b_i$ or
- (b) for some $i, j \in \{0, \dots, k\}$ $a_i > \bar{x}b_i$, $a_j < \bar{x}b_j$, and

$$\frac{\alpha}{\bar{x}} + 2A_S - 2\bar{x}B_S = L \sum_{i=0}^k b_i, \quad (68)$$

where $\alpha \geq 0$,

then the positive equilibrium \bar{x} of (1) is stable on the interval $(0, \infty)$.

When $\sum_{i=0}^k |a_i - \bar{x}b_i| \geq \beta$, the following results show that the positive equilibrium of (1) may be globally asymptotically stable on a subspace of the initial conditions. First we will need the following lemma.

Lemma 15. Let $0 < m \leq \bar{x}$, and for some $N \in \{0, 1, \dots\}$ $x_{N-i} \geq m$, $i = 0, \dots, k$. Suppose that

$$x_{n+1} - \bar{x} = \sum_{i=0}^k h_i (x_{n-i} - \bar{x}), \quad n = 0, 1, \dots, \quad (69)$$

where the nonnegative functions $h_i : [0, \infty)^{k+1} \rightarrow [0, \infty)$. Assume that, for this N , $\sum_{i=0}^k h_i < 1$. Then $x_{N+1} \geq m$.

Proof. Let $K \in \mathbb{R}$. Then (69) has the generalized identity

$$x_{n+1} - \bar{x} - K \sum_{i=0}^k h_i = \sum_{i=0}^k h_i [(x_{n-i} - \bar{x}) - K], \quad (70)$$

$$n = 0, 1, \dots$$

Choose $N \in \{0, 1, \dots\}$. First, suppose that $x_{N-i} \geq \bar{x}$ for $i = 0, \dots, k$. Then by (69)

$$x_{N+1} - \bar{x} = h_0 (x_N - \bar{x}) + \dots + h_k (x_{N-k} - \bar{x}) \geq 0. \quad (71)$$

Thus $x_{N+1} \geq m$.

Second, suppose that $m \leq x_{N-i} < \bar{x}$ for some $i \in \{0, \dots, k\}$. Then $m < \bar{x}$. Let $K = m - \bar{x}$. Then

$$x_{N+1} - \bar{x} - (m - \bar{x}) \sum_{i=0}^k h_i = \sum_{i=0}^k h_i (x_{N-i} - m) \geq 0. \quad (72)$$

By assumption $\sum_{i=0}^k h_i < 1$ for this N and so

$$x_{N+1} - \bar{x} \geq (m - \bar{x}) \sum_{i=0}^k h_i > m - \bar{x}. \quad (73)$$

Thus $x_{N+1} > m$. \square

Theorem 16. Let $\sum_{i=0}^k b_i > 0$ and $m = \min\{\bar{x}, x_0, \dots, x_{-k}\}$. Assume that $0 \leq \sum_{i=0}^k |a_i - \bar{x}b_i| - \beta < m \sum_{i=0}^k b_i$. Then the positive equilibrium of (1) is globally asymptotically stable on the interval $((\sum_{i=0}^k |a_i - \bar{x}b_i| - \beta) / \sum_{i=0}^k b_i, \infty)$.

Proof. Clearly $0 < m$. Since $m \leq x_{-i}$ for $i = 0, \dots, k$, then $m \sum_{i=0}^k b_i \leq \sum_{i=0}^k b_i x_{-i}$ and so

$$\sum_{i=0}^k |g_i| = \sum_{i=0}^k \frac{|a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{-i}} \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta + m \sum_{i=0}^k b_i} < 1 \quad (74)$$

for $n = 0$. Then by Lemma 15 with $h_i = |g_i|$ for $i \in \{0, \dots, k\}$ and $N = 0$ we get that $x_1 \geq m$. Hence $m \leq x_1, \dots, x_{-k}$. Thus

$$\sum_{i=0}^k |g_i| = \sum_{i=0}^k \frac{|a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{1-i}} \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta + m \sum_{i=0}^k b_i} < 1 \quad (75)$$

for $n = 1$. Now applying Lemma 15 again with $h_i = |g_i|$ for $i \in \{0, \dots, k\}$ and $N = 1$ we get that $x_2 \geq m$ and so $m \leq x_2, \dots, x_{-k}$. Thus

$$\sum_{i=0}^k |g_i| = \sum_{i=0}^k \frac{|a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{2-i}} \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta + m \sum_{i=0}^k b_i} < 1 \quad (76)$$

for $n = 2$. Hence by induction we get that

$$\sum_{i=0}^k |g_i| = \sum_{i=0}^k \frac{|a_i - \bar{x}b_i|}{\beta + \sum_{i=0}^k b_i x_{n-i}} \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i|}{\beta + m \sum_{i=0}^k b_i} < 1 \quad (77)$$

for $n = 0, 1, \dots$. Therefore applying Theorem 1 to (24) we have that

$$\lim_{n \rightarrow \infty} y_n = 0 \text{ and so } \lim_{n \rightarrow \infty} x_n = \bar{x} \quad (78)$$

and by applying Theorem 2 to (24) \bar{x} is stable. \square

We now apply Theorem 16 to various cases of (1).

Theorem 17. Let $\sum_{i=0}^k b_i > 0$ and $m = \min\{\bar{x}, x_0, \dots, x_{-k}\}$. Then the positive equilibrium \bar{x} of (1) is globally asymptotically stable on the interval $((\sum_{i=0}^k |a_i - \bar{x}b_i| - \beta) / \sum_{i=0}^k b_i, \infty) \subset [0, \infty)$ provided one of the following holds:

- (1) $a_i = \bar{x}b_i$ for all $i \in \{0, \dots, k\}$;
- (2) $a_i \geq \bar{x}b_i$ for $i \in \{0, \dots, k\}$, $\alpha = 0$, $\beta > 0$, and $m > 0$;
- (3) $a_i \leq \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and either
 - (a) $\beta > 0$, $2\beta^2 + 2\beta \sum_{i=0}^k a_i \leq \alpha \sum_{i=0}^k b_i$ and $(\bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i - \beta) / \sum_{i=0}^k b_i < m$ or
 - (b) $\beta = 0$ and $0 < \alpha / \bar{x} \sum_{i=0}^k b_i < m$;
- (4) for some $i, j \in \{0, \dots, k\}$ $a_i > \bar{x}b_i$, $a_j < \bar{x}b_j$, and $0 \leq \alpha / \bar{x} - 2\beta + 2A_S - 2\bar{x}B_S < m \sum_{i=0}^k b_i$.

Proof. (1) If $\alpha, \beta > 0$, then Theorem 9 part (1) applies. If $\alpha = \beta = 0$, (1) becomes

$$x_{n+1} = \frac{\sum_{i=0}^k a_i x_{n-i}}{\sum_{i=0}^k b_i x_{n-i}} = \frac{\bar{x} \sum_{i=0}^k b_i x_{n-i}}{\sum_{i=0}^k b_i x_{n-i}} = \bar{x}. \quad (79)$$

(2) In this case we have $|a_i - \bar{x}b_i| = a_i - \bar{x}b_i$ for $i \in \{0, \dots, k\}$ and since $\alpha = 0$, then $\beta = \sum_{i=0}^k a_i - \bar{x} \sum_{i=0}^k b_i = \sum_{i=0}^k |a_i - \bar{x}b_i|$. Thus $0 = (\sum_{i=0}^k |a_i - \bar{x}b_i| - \beta) / \sum_{i=0}^k b_i < m$ and the result follows from Theorem 16.

(3) In this case we have $|a_i - \bar{x}b_i| = \bar{x}b_i - a_i$ for $i \in \{0, \dots, k\}$.

- (a) Assume that $\beta > 0$, $2\beta^2 + 2\beta \sum_{i=0}^k a_i \leq \alpha \sum_{i=0}^k b_i$ and $(\bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i - \beta) / \sum_{i=0}^k b_i < m$. By Lemma 8 with $L = 0$, $\beta + \sum_{i=0}^k a_i \leq \bar{x} \sum_{i=0}^k b_i$. Thus

$$0 \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i| - \beta}{\sum_{i=0}^k b_i} < m \quad (80)$$

and so the result follows from Theorem 16.

- (b) Assume that $\beta = 0$ and $0 < \alpha / \bar{x} \sum_{i=0}^k b_i < m$. Then

$$\frac{\alpha}{\bar{x}} = \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i = \sum_{i=0}^k |a_i - \bar{x}b_i|. \quad (81)$$

Thus $0 < \sum_{i=0}^k |a_i - \bar{x}b_i| / \sum_{i=0}^k b_i < m$ and so the result follows from Theorem 16.

(4) In this case from the equilibrium equation we get that

$$\begin{aligned} & \sum_{i=0}^k |a_i - \bar{x}b_i| - \beta \\ &= \bar{x} \sum_{i=0}^k b_i - \sum_{i=0}^k a_i + 2A_S - 2\bar{x}B_S - \beta \\ &= \frac{\alpha}{\bar{x}} - \beta + 2A_S - 2\bar{x}B_S - \beta. \end{aligned} \quad (82)$$

Suppose that $\beta \geq 0$ and $\alpha \geq 0$. Then

$$0 \leq \frac{\sum_{i=0}^k |a_i - \bar{x}b_i| - \beta}{\sum_{i=0}^k b_i} < m \quad (83)$$

and so the result follows from Theorem 16. \square

We illustrate our results with some examples.

Example 18. Equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots, \quad (84)$$

with nonnegative initial conditions and positive coefficients was considered in [4], and it was proved that the condition $A \geq \beta + \gamma + \delta$ implies global asymptotic stability. By Theorem 7 the condition

$$\begin{aligned} & |\beta - B\bar{x}| + |\gamma - C\bar{x}| + \delta < A + (B + C)L, \\ & L = \frac{\min\{\alpha, \beta, \gamma, \delta\}}{\max\{A, B, C\}}, \end{aligned} \quad (85)$$

implies global asymptotic stability of the unique positive equilibrium \bar{x} . If $\bar{x} \leq \beta/B$, $\bar{x} \leq \gamma/C$, then the sufficient condition for global asymptotic stability of the unique positive equilibrium \bar{x} of (84) becomes

$$\beta + \gamma + \delta < A + (B + C)(L + \bar{x}). \quad (86)$$

Example 19. Equation

$$x_{n+1} = \frac{\alpha + x_{n-1}}{A + Bx_n + x_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \dots, \quad (87)$$

with nonnegative initial conditions and positive coefficients was considered in [4], and it was proved that the condition $A \geq 1$ implies global asymptotic stability. Theorem 7 implies that, for the condition

$$|1 - \bar{x}| < A + (B + D)(L - \bar{x}) + L, \quad (88)$$

where

$$L = \frac{\min\{\alpha, 1\}}{\max\{A + (B + D)U, 1\}}, \quad U = \frac{\max\{\alpha, 1\}}{\min\{A, B, D, 1\}}, \quad (89)$$

the positive equilibrium \bar{x} is globally asymptotically stable. If $\bar{x} \leq 1$ then the sufficient condition for global asymptotic stability of the unique positive equilibrium \bar{x} of (87) becomes

$$1 < \bar{x} + A + (B + D)(L - \bar{x}) + L. \quad (90)$$

Example 20. Equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Cx_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \dots, \quad (91)$$

with nonnegative initial conditions and positive coefficients was considered in [4, 10]. In view of Theorem 7 \bar{x} is globally asymptotically stable if the condition

$$|\beta - B\bar{x}| + |\gamma - C\bar{x}| + |\delta - D\bar{x}| < (B + C + D)L, \quad (92)$$

where

$$L = \frac{\min\{\beta, \gamma, \delta\}}{\max\{B, C, D\}}, \quad \bar{x} = \frac{\beta + \gamma + \delta}{B + C + D}, \quad (93)$$

is satisfied. For instance if

$$\frac{\beta}{B} = \frac{\gamma}{C} < \frac{\delta}{D}, \quad (94)$$

then it is clear that $\beta/B = \gamma/C < \bar{x} < \delta/D$ and condition (92) becomes

$$-(\beta - B\bar{x}) - C(\beta - B\bar{x}) + \delta - D\bar{x} < (B + C + D)L \quad (95)$$

or $((C + 1)B - D)\bar{x} < (B + C + D)L + (C + 1)\beta - \delta$.
If

$$\frac{\beta}{B} < \frac{\gamma}{C} < \frac{\delta}{D}, \quad (96)$$

then it is clear that $\beta/B < \bar{x} < \delta/D$ and condition (92) becomes

$$-(\beta - B\bar{x}) + |\beta - B\bar{x}| + \delta - D\bar{x} < (B + C + D)L \quad (97)$$

or $|\beta - B\bar{x}| < \beta - \delta + (D - B)\bar{x} + (B + C + D)L$.

Note that, by Theorem 7, condition (92) also implies that the positive equilibrium \bar{x} is globally asymptotically stable even for the special cases of (91) where exactly one of the coefficients B , C , or D is zero. These three cases were considered in [4] but no global stability or attractivity results were presented.

As it was shown in [10, 11] (91) has a minimal period-two solution for some values of the parameters.

Example 21. Equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots, \quad (98)$$

with nonnegative initial conditions and positive coefficients was considered in [4], but no global stability or attractivity results were presented. In view of Theorem 7, the positive equilibrium \bar{x} is globally asymptotically stable if the condition

$$|\beta - B\bar{x}| + |\gamma - C\bar{x}| + \delta < (B + C)L, \quad (99)$$

where

$$L = \frac{\min\{\alpha, \beta, \gamma, \delta\}}{\max\{B, C\}}, \quad (100)$$

is satisfied.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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